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ELLIPTIC EQUATIONS OF ORDER $2m$ IN ANNULAR DOMAINS

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ABSTRACT. In this paper we study the existence of positive radial solutions for some semilinear elliptic problems of order $2m$ in an annulus with Dirichlet boundary conditions. We consider a nonlinearity which is either sublinear or the sum of a sublinear and a superlinear term.

1. INTRODUCTION

Let $\Omega(a, b)$ denote the annulus $\{x \in \mathbb{R}^n; a < |x| < b\}$, $0 < a < b < \infty$, $n \geq 2$, and consider the semilinear elliptic problems

$$(-1)^m \Delta^m u = g(|x|)f(u) \quad \text{in} \quad \Omega(a, b) \tag{1.1}$$

and

$$(-1)^m \Delta^m u = \lambda g(|x|)f(u) + k(|x|)h(u) \quad \text{in} \quad \Omega(a, b) \tag{1.2}$$

with the boundary conditions

$$u = \frac{\partial u}{\partial \nu} = \cdots = \frac{\partial^{m-1}}{\partial \nu^{m-1}} u = 0 \quad \text{on} \quad \partial\Omega(a, b) \tag{1.3}$$

where $\lambda > 0$ is a parameter, $\frac{\partial}{\partial \nu}$ is the outward normal derivative, m is a positive integer and f, g, h, k satisfy at least the following assumptions:

(H_1) $f, h : [0, \infty) \to [0, \infty)$ are continuous functions;

(H_2) $g, k : [a, b] \to [0, \infty)$ are continuous functions such that g, $k \not\equiv 0$ in $[a, b]$.

When $m = 1$ the existence of a positive radial solution of problem (1.1), (1.3) has been intensively studied in the case where f is superlinear at 0 and ∞ (see *e.g.* [2]–[4], [6], [11], [13]). The approach used in most papers was the shooting method. In contrast the result of [2] was obtained by a variational approach and the use of *a priori* estimates. The case $m \geq 1$ was treated in [8] and [9] using *a priori* estimates and well-known properties of compact mappings taking a cone in a Banach space into itself (see [10]).

When $m = 1$ and f is sublinear at 0 and ∞ problem (1.1), (1.3) possesses at least one positive radial solution. This case was studied in [11] and [15] using the shooting method and the fixed point theorem in cones respectively.

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Finally, when $m = 1$, $g = k = 1$, $\lambda > 0$, $f(u) = u^q$, $0 < q < 1$, and $h(u) = u^p$, $p > 1$, equation (1.2) in a smooth bounded domain with Dirichlet boundary condition was recently studied in [1]. Some results extend to the case where f is concave and behaves like u^q, $0 < q < 1$ near $u = 0$. Also if $1 < p < \frac{n+2}{n-2}$, u^p can be replaced by a function h with the same behavior near $u = 0$ and near $u = \infty$.

In this paper we first prove an existence result for problem (1.1), (1.3) when f is sublinear at 0 and ∞. We do not require any monotonicity assumptions on f. Then we consider problem (1.2), (1.3) when f is sublinear at 0 and h is superlinear at 0 and possibly at ∞. For this problem we also assume that f is nondecreasing.

Our main results are the following two theorems.

Theorem 1.1. *Let f satisfy* (H_1) *and let g satisfy* (H_2). *Assume moreover that the following condition holds*:

(H_3) $\lim_{u\to 0} \frac{f(u)}{u} = \infty$ *and* $\lim_{u\to\infty} \frac{f(u)}{u} = 0$.

Then problem (1.1), (1.3) *has at least one positive radial solution in* $C^{2m}(\overline{\Omega(a\,,\,b)})$.

Theorem 1.2. *Let f, h satisfy* (H_1) *and let g, k satisfy* (H_2). *Assume moreover that the following conditions hold*:

(H_4) $\lim_{u\to 0} \frac{h(u)}{u} = 0$;

(H_5) $\lim_{u\to 0} \frac{f(u)}{u} = \infty$;

(H_6) *f is nondecreasing.*

Then there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ problem (1.2), (1.3) *has at least one positive radial solution in* $C^{2m}(\overline{\Omega(a\,,\,b)})$.

Since we are interested in positive radial solutions, problems (1.1), (1.3) and (1.2), (1.3) reduce to the one-dimensional boundary value problems

$$(-1)^m\Delta^m u(t) = g(t)f(u(t)), \qquad t \in (a\,,\,b), \tag{1.4}$$

and

$$(-1)^m\Delta^m u(t) = \lambda g(t)f(u(t)) + k(t)h(u(t)), \qquad t \in (a\,,\,b), \tag{1.5}$$

with the boundary conditions

$$u^{(j)}(a) = u^{(j)}(b) = 0, \qquad j = 0, \ldots, m-1, \tag{1.6}$$

where Δ denotes the polar form of the Laplacian, *i.e.*:

$$\Delta = t^{1-n}\frac{d}{dt}(t^{n-1}\frac{d}{dt}).$$

The proofs make use of some precise estimates for the Green's function of the corresponding linear two-point boundary value problem. The other tools are a fixed point theorem in cones and the Schauder fixed point theorem.

In Section 2 we give some simple inequalities of the Green's function. Various results concerning disconjugate operators are needed. We also give *a priori* bounds for positive solutions of problem (1.5), (1.6) when h is superlinear at

∞. In Section 3 we prove Theorem 1.1. In Section 4 we prove Theorem 1.2 and we give a bound for λ_0 when in addition h is superlinear at ∞.

2. Preliminaries

The homogeneous Dirichlet problem

$$\begin{cases} \Delta^m v = 0 & \text{in } [a, b], \\ v^{(j)}(a) = v^{(j)}(b) = 0, & j = 0, \ldots, m-1, \end{cases}$$

has only the trivial solution. Then it is well-known (see *e.g.* [14], p. 29) that the operator $(-1)^m \Delta^m$ with Dirichlet boundary conditions has one and only one Green's function $G_m(t, s)$.

Theorem 2.1. $G_m(t, s) > 0$ *for* $a < t, s < b$.

Proof. Since $(-1)^m \Delta^m$ is a disconjugate operator on $[a, b]$, this is an immediate consequence of a theorem obtained in [7] (Theorem 11 on p. 108).

As we shall see in section 3 our next result provides very useful estimates for the norm of the integral operator associated with problem (1.4), (1.6).

Theorem 2.2. (i) *There exists a positive constant* C_m *such that*

$$0 \le G_m(t, s) \le C_m (s-a)^m (b-s)^m, \qquad a \le t, s \le b.$$

(ii) *For any* $\delta \in (0, (b-a)/2)$ *there exists* $\eta \in (0, 1)$ *such that*

$$G_m(t, s) \ge \eta C_m (s-a)^m (b-s)^m, \qquad a \le s \le b \quad \text{and} \quad a + \delta \le t \le b - \delta.$$

In order to prove Theorem 2.2 we shall need some results obtained in [5]. Denote by $\Delta^\star$ the adjoint of Δ.

Let $v, v^\star, w, w^\star \in C^{2m}([a, b])$ be defined by the following relations:

$$\text{(2.1)} \qquad \begin{cases} \Delta^m v = (\Delta^\star)^m v^\star = 0 & \text{in } [a, b], \\ v^{(j)}(a) = v^{\star(j)}(b) = 0, & j = 0, \ldots, m-1, \\ v^{(j)}(b) = v^{\star(j)}(a) = 0, & j = 0, \ldots, m-2 \text{ (if } m \ge 2), \\ v^{(m-1)}(b) = (-1)^{m-1}, \ v^{\star(m-1)}(a) = 1, \end{cases}$$

and

$$\text{(2.2)} \qquad \begin{cases} \Delta^m w = (\Delta^\star)^m w^\star = 0, & \text{in } [a, b], \\ w^{(j)}(a) = w^{\star(j)}(b) = 0, & j = 0, \ldots, m-2 \text{ (if } m \ge 2), \\ w^{(j)}(b) = w^{\star(j)}(a) = 0, & j = 0, \ldots, m-1, \\ w^{(m-1)}(a) = 1, \ w^{\star(m-1)}(b) = (-1)^{m-1}. \end{cases}$$

The functions defined in (2.1), (2.2) are positive on (a, b) because of the disconjugacy of the operators Δ^m and $(\Delta^\star)^m$. Now define

$$K_m(t, s) = \begin{cases} \dfrac{G_m(t, s)}{v(t) v^\star(s)} & \text{on } a < t \le s < b \\ \dfrac{(-1)^m}{v^{\star(m)}(b)} & \text{on } t = a \text{ or } s = b \end{cases}$$

and

$$L_m(t,s)=\begin{cases}\dfrac{G_m(t,s)}{w(t)w^\star(s)} & \text{on} \quad a<s\le t<b\\[2mm] \dfrac{1}{w^{\star(m)}(a)} & \text{on} \quad s=a \text{ or } t=b.\end{cases}$$

Denote by $T_u=\{(t,s)\in[a,b]\times[a,b];\, t\le s\}$ the upper triangle and by $T_l=\{(t,s)\in[a,b]\times[a,b];\, s\le t\}$ the lower triangle. The proof of the following lemma can be found in [5], section 3.

Lemma 2.1. (i) *K_1 is a positive constant on T_u and L_1 is a positive constant on T_l.*

(iia) *If $m\ge 2$ K_m is bounded on T_u and K_m is continuous and positive on $T_u\setminus\{(a,a),(b,b)\}$.*

(iib) *If $m\ge 2$ L_m is bounded on T_l and L_m is continuous and positive on $T_l\setminus\{(a,a),(b,b)\}$.*

Proof of Theorem 2.2. Since Δ^m and $(\Delta^\star)^m$ are disconjugate operators on $[a,b]$ there exist α, $\alpha^\star$, β and $\beta^\star$ in $C^1([a,b])$ such that

$$v(t)=(t-a)^m(b-t)^{m-1}\alpha(t),\quad a\le t\le b,$$
$$v^\star(s)=(s-a)^{m-1}(b-s)^m\alpha^\star(s),\quad a\le s\le b,$$
$$w(t)=(t-a)^{m-1}(b-t)^m\beta(t),\quad a\le t\le b,$$
$$w^\star(s)=(s-a)^m(b-s)^{m-1}\beta^\star(s),\quad a\le s\le b,$$

and

$$\alpha,\ \alpha^\star,\ \beta \text{ and } \beta^\star>0 \text{ on } [a,b].$$

(i) By virtue of Lemma 2.1 we can define

$$M_m=\max\Big(\max_{(t,s)\in T_u}K_m(t,s),\ \max_{(t,s)\in T_l}L_m(t,s)\Big).$$

Then using Theorem 2.1 we get

$$0\le G_m(t,s)\le M_m\|\alpha\|_\infty\|\alpha^\star\|_\infty(t-a)^m(b-t)^{m-1}(s-a)^{m-1}(b-s)^m$$

for $(t,s)\in T_u$ and

$$0\le G_m(t,s)\le M_m\|\beta\|_\infty\|\beta^\star\|_\infty(s-a)^m(b-s)^{m-1}(t-a)^{m-1}(b-t)^m$$

for $(t,s)\in T_l$ and (i) follows with

$$C_m=M_m\max(\|\alpha\|_\infty\|\alpha^\star\|_\infty,\ \|\beta\|_\infty\|\beta^\star\|_\infty)(b-a)^{2(m-1)}.$$

(ii) Let $\delta\in(0,(b-a)/2)$. By Lemma 2.1 we can define

$$A_\delta=\min\Big(\min_{\substack{(t,s)\in T_u\\ a+\delta\le t\le b-\delta}}K_m(t,s),\ \min_{\substack{(t,s)\in T_l\\ a+\delta\le t\le b-\delta}}L_m(t,s)\Big)$$

and $A_\delta>0$. Therefore if $t\in[a+\delta,b-\delta]$ and $s\in[a,b]$ we obtain

$$\begin{aligned}G_m(t,s)&\ge A_\delta\begin{cases}v(t)v^\star(s), & t\le s,\\ w(t)w^\star(s), & s\le t,\end{cases}\\ &\ge CA_\delta\begin{cases}(t-a)^m(b-t)^{m-1}(s-a)^{m-1}(b-s)^m, & t\le s,\\ (s-a)^m(b-s)^{m-1}(t-a)^{m-1}(b-t)^m, & s\le t,\end{cases}\\ &\ge CA_\delta\delta^{2m-1}(b-a)^{-1}(s-a)^m(b-s)^m\end{aligned}$$

for some positive constant C and (ii) is proved.

Now we give an example.

Example. When $m = 1$ the Green's function $G_1(t, s)$ is easily obtained. We have

$$G_1(t, s) = \frac{st^{2-n}}{(n-2)(b^{n-2} - a^{n-2})} \begin{cases} (t^{n-2} - a^{n-2})(b^{n-2} - s^{n-2}), & a \le t \le s \le b, \\ (s^{n-2} - a^{n-2})(b^{n-2} - t^{n-2}), & a \le s \le t \le b, \end{cases}$$

if $n \ge 3$ and

$$G_1(t, s) = \frac{s}{\ln b - \ln a} \begin{cases} (\ln t - \ln a)(\ln b - \ln s), & a \le t \le s \le b, \\ (\ln s - \ln a)(\ln b - \ln t), & a \le s \le t \le b, \end{cases}$$

if $n = 2$.

We conclude this section with the following result.

Theorem 2.3. *Assume* (H_1) *and* (H_2). *Suppose in addition that* h *satisfies the following condition:*

(H_7) $\lim_{u \to \infty} \frac{h(u)}{u} = \infty$.

Then there exist M, M', $M'' > 0$ *such that*

$$\|u\|_\infty \le M \quad \text{and} \quad \|u'\|_\infty \le M'\lambda + M''$$

for all positive solutions $u \in C^{2m}([a, b])$ *of* (1.5), (1.6) *where* M, M' *and* M'' *are independent of* $\lambda > 0$.

Proof. Define

$$\rho(t) = (t-a)^m (b-t)^m, \quad a \le t \le b. \tag{2.3}$$

Let $\varphi \in C^{2m}([a, b])$ be the solution of the boundary value problem

$$\begin{cases} (-1)^m \Delta^m \varphi = k\rho & \text{in } [a, b], \\ \varphi^{(j)}(a) = \varphi^{(j)}(b) = 0, & j = 0, \dots, m-1. \end{cases}$$

By (H_2) and Theorem 2.1 $\varphi > 0$ on (a, b). Then using a proposition obtained in [7] (Proposition 13 on p. 109) we deduce that

$$\varphi^{(m)}(a) > 0 \quad \text{and} \quad (-1)^m \varphi^{(m)}(b) > 0.$$

Therefore there exist c_1, $c_2 > 0$ such that

$$c_1 \rho \le \varphi \le c_2 \rho \quad \text{on} \quad [a, b]. \tag{2.4}$$

By (H_7) there exist $\mu > c_1^{-1}$ and a positive constant c_3 such that

$$h(u) \ge \mu u - c_3 \qquad \text{for } u \ge 0. \tag{2.5}$$

Now let $u \in C^{2m}([a, b])$ be a positive solution of (1.5), (1.6) where $\lambda > 0$. If we multiply equation (1.5) by $t^{n-1}\varphi$ and integrate by parts $2m$ times we obtain

$$\int_a^b t^{n-1} \rho k u \, dt = \int_a^b t^{n-1} \varphi (\lambda g f(u) + k h(u)) \, dt. \tag{2.6}$$

From (2.4)–(2.6) we deduce that

$$\int_a^b t^{n-1}\rho k u\,dt \geq \mu \int_a^b t^{n-1}\varphi k u\,dt - c_4 \geq \mu c_1 \int_a^b t^{n-1}\rho k u\,dt - c_4$$

for some positive constant c_4, hence

$$\int_a^b t^{n-1}\rho k u\,dt \leq \frac{c_4}{\mu c_1 - 1}. \tag{2.7}$$

Using Theorem 2.2(i), (2.4), (2.6) and (2.7) we get

$$u(t) = \int_a^b G_m(t, s)(\lambda g(s) f(u(s)) + k(s)h(u(s)))\,ds \leq M, \quad a \leq t \leq b,$$

for some positive constant M independent of $\lambda > 0$. This gives the first estimate.

Now, if $m = 1$ we can write

$$t^{n-1}u'(t) = -\int_c^t s^{n-1}(\lambda g(s) f(u(s)) + k(s)h(u(s)))\,ds \tag{2.8}$$

for $t \in [a, b]$ with $c \in (a, b)$ such that $u'(c) = 0$. When $m \geq 2$ we have

$$u'(t) = \int_a^b \frac{\partial}{\partial t} G_m(t, s)(\lambda g(s) f(u(s)) + k(s)h(u(s)))\,ds. \tag{2.9}$$

Therefore the second estimate follows from (2.8), (2.9) and the *a priori* L^∞ bound already obtained for u.

Remark 1. Note that in the proof of Theorem 2.3 the condition $g \not\equiv 0$ is not needed and when $g \equiv 0$ in $[a, b]$ we can take $M' = 0$.

3. Proof of Theorem 1.1

As noted in the introduction it is enough to show that problem (1.4), (1.6) has at least one positive solution $u \in C^{2m}([a, b])$. The proof makes use of the following fixed point theorem due to Krasnosel'skii ([12]).

Theorem 3.1. *Let X be a Banach space, K a cone in X and $0 < r < R$. Let $T : \{u \in K; 0 < r \leq ||u|| \leq R\} \to K$ be a compact operator such that $||Tu|| \geq r$ for $||u|| = r$ and $||Tu|| \leq R$ for $||u|| = R$.*
Then T has a fixed point in $\{u \in K; 0 < r \leq ||u|| \leq R\}$.

Now by (H_2) there exists $\delta \in (0, (b-a)/2)$ such that $g \not\equiv 0$ in $[a+\delta, b-\delta]$. Let η be as in Theorem 2.2(ii). Let X be the Banach space $C([a, b])$ endowed with the sup norm and define the cone

$$K = \{u \in X; u \geq 0, \ \min\{u(t); a + \delta \leq t \leq b - \delta\} \geq \eta ||u||_\infty\}.$$

For $u \in K$ we define

$$Tu(t) = \int_a^b G_m(t, s) g(s) f(u(s))\,ds, \qquad a \leq t \leq b.$$

We first show that $TK \subset K$. By Theorem 2.2(i) we have

$$||Tu||_\infty \leq C_m \int_a^b \rho(s) g(s) f(u(s))\,ds \tag{3.2}$$

where ρ is defined by (2.3). Using Theorem 2.2(ii) we obtain

$$\min\{Tu(t)\,;\, a+\delta \le t \le b-\delta\} \ge \eta C_m \int_a^b \rho(s)g(s)f(u(s))\,ds\,. \tag{3.3}$$

From (3.2) and (3.3) we deduce that

$$\min\{Tu(t)\,;\, a+\delta \le t \le b-\delta\} \ge \eta \|Tu\|_\infty .$$

Since by Theorem 2.1 $Tu \ge 0$ we conclude that $TK \subset K$. It is well-known that $T : K \to K$ is completely continuous.

By (H_3) there exists $r > 0$ such that

$$f(u) \ge Cu \qquad \text{for} \quad 0 \le u \le r$$

where C is a positive constant satisfying

$$C\eta \int_{a+\delta}^{b-\delta} G_m\Big(\frac{a+b}{2}, s\Big)g(s)\,ds \ge 1.$$

Now let $u \in K$ be such that $\|u\|_\infty = r$. We have

$$\begin{aligned}
T(u)\Big(\frac{a+b}{2}\Big) &= \int_a^b G_m\Big(\frac{a+b}{2}, s\Big)g(s)f(u(s))\,ds \\
&\ge \int_{a+\delta}^{b-\delta} G_m\Big(\frac{a+b}{2}, s\Big)g(s)f(u(s))\,ds \\
&\ge \Big(C\eta \int_{a+\delta}^{b-\delta} G_m\Big(\frac{a+b}{2}, s\Big)g(s)\,ds\Big)r \\
&\ge r
\end{aligned}$$

which implies that $\|Tu\|_\infty \ge r$.

By (H_3) there exists $r' > 0$ such that

$$f(u) \le C'u \qquad \text{for} \quad u \ge r'$$

where C' is a positive constant satisfying

$$C'C_m \int_a^b \rho(s)g(s)\,ds \le 1.$$

Suppose first that f is bounded. Then there exists $B > 0$ such that $f(u) \le B$ for $u \ge 0$. Then choose $R > r$ such that

$$BC_m \int_a^b \rho(s)g(s)\,ds \le R.$$

Let $u \in K$ be such that $\|u\|_\infty = R$. By (3.2) we have

$$\begin{aligned}
\|Tu\|_\infty &\le C_m \int_a^b \rho(s)g(s)f(u(s))\,ds \\
&\le BC_m \int_a^b \rho(s)g(s)\,ds \\
&\le R.
\end{aligned}$$

Now if f is unbounded, we choose R such that $R > \max\{r, r'\}$ and $f(u) \leq f(R)$ for $0 \leq u \leq R$. Let $u \in K$ be such that $\|u\|_\infty = R$. By (3.2) we have

$$\begin{aligned}\|Tu\|_\infty \leq & C_m \int_a^b \rho(s)g(s)f(u(s))\,ds \\ \leq & (C'C_m \int_a^b \rho(s)g(s)\,ds)R \\ \leq & R.\end{aligned}$$

Therefore in both cases we get $\|Tu\|_\infty \leq R$ for $u \in K$ such that $\|u\|_\infty = R$.

Thus we may apply Theorem 3.1 to conclude that T has a fixed point in $\{u \in K ; 0 < r \leq \|u\|_\infty \leq R\}$. By Theorem 2.1, (H_1), (H_2) and the properties of the Green's function any nontrivial fixed point of T in K yields a positive solution of problem (1.4), (1.6) in $C^{2m}([a, b])$. The proof of the theorem is complete.

Remark 2. Theorem 3.1 still holds if both inequalities are reversed. Then, using analogous arguments, we could treat the case where f is superlinear at 0 and ∞. As noted in the introduction a different proof of the superlinear case was given in [9].

Remark 3. Clearly Theorem 1.1 remains true for a nonlinearity $f(|x|, u)$ satisfying:

(i) $f : [a, b] \times [0, \infty) \to [0, \infty)$ is a continuous function;

(ii) $\lim_{u\to 0} \min_{t\in[a,b]} \frac{f(t, u)}{u} = \infty$ and $\lim_{u\to\infty} \max_{t\in[a,b]} \frac{f(t, u)}{u} = 0$.

4. Proof of Theorem 1.2

Again it is enough to show that there exists $\lambda_0 > 0$ such that for all $\lambda \in (0, \lambda_0)$ problem (1.5), (1.6) has at least one positive solution $u \in C^{2m}([a, b])$. We begin with a lemma.

Lemma 4.1. *Let $N > 0$. For all $R > \frac{3\|g\|_\infty f(0)}{N}$ we can find $\lambda_0 > 0$ (depending on R and N) such that for all $\lambda \in (0, \lambda_0]$ and $u \in [0, R\lambda]$ we have*

$$NR\lambda \geq \lambda\|g\|_\infty f(u) + \|k\|_\infty h(u).$$

Proof. Since $\|g\|_\infty f(0) \leq \frac{NR}{3}$, there exists $\lambda_1 > 0$ such that

$$\|g\|_\infty f(u) \leq \frac{NR}{2} \qquad \text{for } u \in [0, R\lambda_1].$$

Let $\varepsilon \in (0, \frac{N}{2\|k\|_\infty}]$. By (H_4) there exists $r > 0$ such that $h(u) \leq \varepsilon u$ for $u \in [0, r]$. Define $\lambda_0 = \min(\frac{r}{R}, \lambda_1)$ and let $\lambda \in (0, \lambda_0]$ and $u \in [0, R\lambda]$. Then we have

$$\lambda\|g\|_\infty f(u) + \|k\|_\infty h(u) \leq \frac{NR\lambda}{2} + \|k\|_\infty \varepsilon u \leq NR\lambda.$$

The proof of the lemma is complete.

Now let

$$N = (C_m \int_a^b \rho(s)ds)^{-1}$$

where C_m is given by Theorem 2.2(i) and ρ is defined by (2.3). Fix $R > \frac{3\|g\|_\infty f(0)}{N}$ and let λ_0 be as in Lemma 4.1. Fix $\lambda \in (0, \lambda_0]$. First consider the solution $\psi \in C^{2m}([a, b])$ of the boundary value problem

$$\begin{cases} (-1)^m \Delta^m \psi = g\rho & \text{in } [a, b], \\ \psi^{(j)}(a) = \psi^{(j)}(b) = 0, & j = 0, \dots, m-1. \end{cases} \tag{4.1}$$

As in the proof of theorem 2.3 $\psi > 0$ on (a, b), $\psi^{(m)}(a) > 0$, $(-1)^m \psi^{(m)}(b) > 0$ and there exist $d_1, d_2 > 0$ such that

$$d_1 \rho \le \psi \le d_2 \rho \quad \text{on} \quad [a, b]. \tag{4.2}$$

By (H_5) we can choose $r = r(\lambda) \in (0, R\lambda]$ such that

$$f(u) \ge \frac{1}{\lambda d_1} u \qquad \text{for } 0 \le u \le r \tag{4.3}$$

where d_1 is given by (4.2). Let $c > 0$ be such that

$$c(b-a)^{2m} \le r \tag{4.4}$$

and consider the set of functions

$$Z = \{u \in C([a, b]);\ c\rho(t) \le u(t) \le R\lambda,\ a \le t \le b\}.$$

Clearly, Z is a nonempty closed bounded convex subset of $C([a, b])$ equipped with the sup norm. For $u \in Z$ we define

$$F(u(t)) = \int_a^b G_m(t, s)(\lambda g(s) f(u(s)) + k(s) h(u(s)))\, ds$$

for $a \le t \le b$. We first prove that $FZ \subset Z$. Indeed let $u \in Z$. By Theorem 2.2(i) we have

$$\|F(u)\|_\infty \le C_m \int_a^b \rho(s)(\lambda g(s) f(u(s)) + k(s) h(u(s)))\, ds. \tag{4.5}$$

Using (4.5) and Lemma 4.1 we get

$$\begin{aligned} \|F(u)\|_\infty \le & C_m \int_a^b \rho(s)(\lambda \|g\|_\infty f(u(s)) + \|k\|_\infty h(u(s)))\, ds \\ \le & R\lambda. \end{aligned}$$

Now by virtue of (H_6), (4.3) and (4.4) we have

$$\begin{aligned} F(u(t)) \ge & \lambda \int_a^b G_m(t, s) g(s) f(u(s))\, ds \\ \ge & \lambda \int_a^b G_m(t, s) g(s) f(c\rho(s))\, ds \\ \ge & c d_1^{-1} \int_a^b G_m(t, s) g(s) \rho(s)\, ds \\ \ge & c\rho(t) \end{aligned}$$

for $t \in [a, b]$ because the solution ψ of (4.1) is given by

$$\psi(t) = \int_a^b G_m(t, s) g(s) \rho(s)\, ds.$$

Therefore $FZ \subset Z$. Since F is compact, the Schauder fixed point theorem implies that F has a fixed point $u \in Z$. By the properties of the Green's function any fixed point of F in Z yields a positive solution of problem (1.5), (1.6) in $C^{2m}([a, b])$. The theorem is proved.

Now we shall show that if in addition h is superlinear at ∞ we can give a bound for λ_0. Let us define

$$A = \{\mu > 0;\ (1.5), (1.6) \text{ has a positive solution for all } \lambda \in (0, \mu)\}.$$

By Theorem 1.2 $A \neq \varnothing$. Thus, if we define

$$\lambda^\star = \sup A$$

we have $\lambda^\star \in (0, \infty]$.

Lemma 4.2. *Assume moreover that h satisfies* (H_7). *Then* $\lambda^\star < \infty$.

Proof. By Theorem 2.3 there exists $M > 0$ such that for all $\lambda > 0$ and all positive solutions $u \in C^{2m}([a, b])$ of problem (1.5), (1.6) we have

$$\|u\|_\infty \leq M. \tag{4.6}$$

(H_1), (H_5) and (H_6) imply that there exists $\overline{\lambda} > 0$ such that

$$u < \overline{\lambda} f(u) \quad \forall\, u \in (0, M].$$

Therefore we obtain

$$g(s) u < \overline{\lambda} g(s) f(u) \tag{4.7}$$

for $u \in (0, M]$ and $s \in [a, b]$ such that $g(s) \neq 0$. Now let $\psi \in C^{2m}([a, b])$ be as in the proof of Theorem 1.2 and let $\lambda > 0$ be such that (1.5), (1.6) has a positive solution $u \in C^{2m}([a, b])$. Multiplying (1.5) by $t^{n-1}\psi$ and integrating by parts $2m$ times we obtain

$$\int_a^b s^{n-1} \psi(s)(\lambda g(s) f(u(s)) + k(s) h(u(s)))\, ds = \int_a^b s^{n-1} g(s) \rho(s) u(s)\, ds.$$

Since by (H_2), (4.2), (4.6) and (4.7) we have

$$\int_a^b s^{n-1} g(s) \rho(s) u(s)\, ds < \overline{\lambda} d_1^{-1} \int_a^b s^{n-1} \psi(s) g(s) f(u(s))\, ds$$

we deduce that $\lambda < \overline{\lambda} d_1^{-1}$, hence $\lambda^\star \leq \overline{\lambda} d_1^{-1}$. The proof of the lemma is complete.

We conclude this section with a result concerning the limit case $\lambda = \lambda^\star < \infty$.

Theorem 4.1. *Assume* (H_1) *and* (H_2). *Suppose in addition that h satisfies* (H_7) *and that* $f(0) > 0$. *Let* $\widehat{\lambda} \in (0, \infty)$ *be such that for all* $\lambda \in (0, \widehat{\lambda})$ *problem* (1.5), (1.6) *has a positive solution* $u \in C^{2m}([a, b])$. *Then for* $\lambda = \widehat{\lambda}$ *problem* (1.5), (1.6) *has at least one positive solution in* $C^{2m}([a, b])$.

Proof. Let (λ_n) be a sequence in $(0, \widehat{\lambda})$ such that $\lambda_n \to \widehat{\lambda}$. By our assumption for each $n \in \mathbb{N}$ there exists a positive solution $u_n \in C^{2m}([a, b])$ of problem

(1.5), (1.6). By Theorem 2.3 (u_n) is bounded in the sup norm. Since (u'_n) is also bounded in the sup norm we deduce that (u_n) is equicontinuous. By virtue of the Ascoli theorem there is a subsequence (u_{n_k}) of (u_n) which converges uniformly to a function $u \in C([a, b])$ such that $u \geq 0$. Clearly

$$u(t) = \int_a^b G_m(t, s)(\widehat{\lambda} g(s) f(u(s)) + k(s) h(u(s)))\, ds \tag{4.8}$$

for $t \in [a, b]$. Thus $u \in C^{2m}([a, b])$ and u is a solution of problem (1.5), (1.6). Since $f(0) > 0$, Theorem 2.1, (H_2) and (4.8) imply that $u(t) > 0$ for $t \in (a, b)$. The proof of the theorem is complete.

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